

Impact of anti-virus software on computer virus dynamical behavior

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The impact of anti-virus software on the spreading of computer virus is investigated via developing a mathematical model in this paper. Considering the anti-virus software may not be effective, as it may be an outdated version, and then the computers may be infected with a reduced incidence rate. According to the method of next generation matrix, the basic reproduction number is derived. By introducing appropriate Lyapunov function and the Routh stability criterion, acquiring the stability conditions of the virus-free equilibrium and virus equilibrium. The effect of anti-virus software and disconnecting rate on the spreading of virus are also analyzed. When combined with the numerical results, a set of suggestions are put forward for eradicating virus effectively.

Keywords: Anti-virus software; computer virus; basic reproduction number; Routh stability criterion.

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1. Introduction

A computer virus is a malicious program that can replicate itself and spread from computer to computer. With the advance of computer networks and communication technologies, the number and sort of computer viruses have increased dramatically, which causes huge losses to the human society. Consequently, establishing reasonable computer virus propagation models by considering the characteristics of computer virus, and by model analysis, understanding the spread law of the virus over the network, are currently a hot topic of research.¹⁻⁵

One of the significant features of computer virus is its latent characteristic,⁶ which means that when virus enter a host computer, they hide themselves and only

breaking out after a certain period. The virus in both latency and breaking states have infectivity. However, the virus in latency is perceived hardly by computer users, and then it can infect more computers. While the breaking virus will be easily perceived, so computer users will clear virus by running anti-virus software immediately. Consequently, latent computers would be cured at a lower rate than breaking computers.⁷

One common feature is shared by a computer virus and a biological virus is infectivity. Based on this fact, some classic epidemic models, such as the SIRS model, SEIR model, SEIRS model, are usually borrowed to depict the spread of computer virus.⁸⁻¹⁰ In recent times, research attentions have been rewarded to improve the combination of virus propagation model and anti-virus countermeasures to study the prevalence of virus.¹¹⁻¹³

Considering the anti-virus software may not be effective and the computer virus may hide themselves, in this paper, a novel computer virus spreading model is established. The infected computers are divided into two types, i.e. latent computers and breaking computers. At the same time, the effect of anti-virus software on virus spreading is considered. By introducing appropriate Lyapunov function, it is proved that the virus-free equilibrium is globally asymptotically stable if $R_0 \leq 1$. According to the Routh stability criterion, the condition that the virus equilibrium is locally asymptotically stable is obtained. Finally, a set of suggestions are recommended for eradicating virus effectively.

2. SVLB Model

All computers connected to the network are divided into four classes: susceptible computers without anti-virus software (S), susceptible computers with anti-virus software (V), infected computers that are currently latent (L) and infected computers that are currently breaking out (B). Let $S(t), V(t), L(t), B(t)$ denote their corresponding percentages of all computers at time t , respectively. Then

$$S(t) + V(t) + L(t) + B(t) = 1.$$

The basic assumptions and the parameters in concern with our model are presented below:

- (i) All newly connected computers are all virus-free.
- (ii) External computers are connected to the Internet at positive constant rate μ , and internal computers are disconnected from the Internet are also at this rate. $q_1(q_2)$ is the rate of computers without (with) anti-virus software before they are connected to the Internet and then $q_1 + q_2 = 1$.
- (iii) Each virus-free computer gets contact with an infected computer at a bilinear incidence rate $\beta S(L + B)$, where β is a positive constant.
- (iv) p is the rate of susceptible computers with anti-virus at positive. The model also assumes that the anti-virus software may not be too much effective as it may be

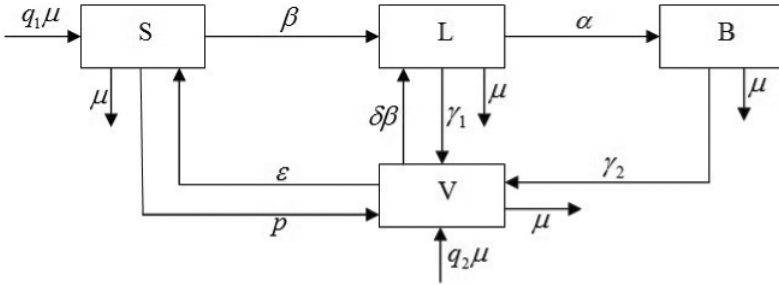


Fig. 1. Schematic diagram for the SVLB model.

an outdated version or it may not be updated. In this case, ε is the rate of migration of computers from class V to class S .

- (v) Assume that while a computer with anti-virus software, they can still re-acquire the virus with reduced incidence rate $\delta\beta$, where $0 < \delta < 1$.
- (vi) Latent computers break out at positive constant rate α .
- (vii) Infected computers are cured by installing anti-virus or updated anti-virus software and then transmit from class L or B to class V . The cure rate of latent and breaking computers is γ_1 and γ_2 and $0 < \gamma_1 < \gamma_2$.

The transfer diagram is depicted in Fig. 1. Then, the computer virus spreading model can be expressed by the following differential equations:

$$\begin{cases} \dot{S} = q_1\mu - \beta S(L + B) - (\mu + p)S + \varepsilon V, \\ \dot{V} = q_2\mu + pS - \delta\beta V(L + B) - (\mu + \varepsilon)V + \gamma_1 L + \gamma_2 B, \\ \dot{L} = \beta(L + B)(S + \delta V) - (\gamma_1 + \alpha + \mu)L, \\ \dot{B} = \alpha L - (\gamma_2 + \mu)B. \end{cases} \quad (1)$$

The feasible region for system (1) is $\Omega = \{(S, L, V, B) \in \mathbb{R}_+^4 | S + L + V + B = 1\}$.

3. The Equilibrium and Basic Reproduction Number

Clearly, system (1) has a virus-free equilibrium:

$$P_0(S_0, V_0, L_0, B_0) = \left(\frac{\varepsilon + q_1\mu}{\mu + \varepsilon + p}, \frac{p + q_2\mu}{\mu + \varepsilon + p}, 0, 0 \right) \quad (2)$$

and a viral equilibrium:

$$P^*(S^*, V^*, B^*, L^*) = (R_5 - \delta T_1, T_1, (\gamma_2 + \mu)(R_3 - R_4 T_1), \alpha(R_3 - R_4 T_1)), \quad (3)$$

where

$$R_1 = \frac{(\beta - \mu - p)\delta - \varepsilon}{\beta\delta(1 - \delta)} + \frac{(\gamma_1 + \alpha + \mu)(\gamma_2 + \mu)}{\beta(1 - \delta)(\gamma_2 + \alpha + \mu)},$$

$$R_2 = \frac{q_1\mu}{\beta\delta(1-\delta)} - \frac{(\gamma_1 + \alpha + \mu)^2(\gamma_2 + \mu)^2}{\beta^2\delta(1-\delta)(\gamma_2 + \alpha + \mu)^2} - \frac{(\gamma_1 + \alpha + \mu)(\gamma_2 + \mu)(\mu + p + \beta)}{\beta^2\delta(1-\delta)(\gamma_2 + \alpha + \mu)},$$

$$R_3 = \frac{[\beta(\gamma_2 + \alpha + \mu) - (\gamma_1 + \alpha + \mu)(\gamma_2 + \mu)]}{\beta(\gamma_2 + \alpha + \mu)^2},$$

$$R_4 = \frac{1 - \delta}{\gamma_2 + \alpha + \mu}, \quad R_5 = \frac{(\gamma_1 + \alpha + \mu)(\gamma_2 + \mu)}{\beta(\gamma_2 + \alpha + \mu)}, \quad T_1 = \frac{R_1 - \sqrt{R_1^2 - 4R_2}}{2}.$$

According to the next generation matrix,¹⁴ one can derive the basic reproduction number of model (1)

$$R_0 = \frac{a_1(\gamma_2 + \mu + \alpha)(\beta S_0 + \delta\beta V_0)}{(\gamma_1 + \alpha + \mu)(\gamma_2 + \mu)}, \tag{4}$$

where $a_1 = \frac{(\gamma_1 + \alpha + \mu)(\gamma_2 + \mu)}{\alpha(\gamma_2 V_0 + \mu) + (\gamma_2 + \mu)(\gamma_1 V_0 + \mu)}$.

4. Stability Analysis

4.1. The virus-free equilibrium globally stable

Theorem 1. *The virus-free equilibrium P_0 is globally stable with respect to Ω if $R_0 < 1$.*

Proof. For the virus-free equilibrium $P_0 = (S_0, V_0, 0, 0)$, S_0 and V_0 satisfies equations:

$$\begin{cases} q_1\mu - (\mu + p)S_0 + \varepsilon V_0 = 0, \\ q_2\mu + pS_0 - (\mu + \varepsilon)V_0 = 0. \end{cases} \tag{5}$$

Then, Eq. (1) can be rewritten as follows:

$$\begin{cases} \dot{S} = S \left[q_1\mu \left(\frac{1}{S} - \frac{1}{S_0} \right) + \varepsilon \left(\frac{V}{S} - \frac{V_0}{S_0} \right) - \beta(L + B) \right], \\ \dot{V} = V \left[q_2\mu \left(\frac{1}{V} - \frac{1}{V_0} \right) + p \left(\frac{S}{V} - \frac{S_0}{V_0} \right) - \delta\beta(L + B) \right] + \gamma_1 L + \gamma_2 B, \\ \dot{L} = \beta(L + B)(S + \delta V) - (\gamma_1 + \alpha + \mu)L, \\ \dot{B} = \alpha L - (\gamma_2 + \mu)B. \end{cases} \tag{6}$$

Define the Lyapunov function

$$G(t) = a_1 \int_{S_0}^S \frac{x - S_0}{x} dx + a_2 \int_{V_0}^V \frac{x - V_0}{x} dx + a_3 L + a_4 B, \tag{7}$$

where $a_1 = \frac{(\gamma_1 + \alpha + \mu)(\gamma_2 + \mu)}{\alpha(\gamma_2 V_0 + \mu) + (\gamma_2 + \mu)(\gamma_1 V_0 + \mu)}$ and $a_2, a_3, a_4 \geq 0$ will be determined later. Then, differentiating G with respect to t along solutions of Eq. (6) gives

$$\begin{aligned} \dot{G}(S, V, L, B) &= a_1 \left(1 - \frac{S_0}{S}\right) \dot{S} + a_2 \left(1 - \frac{V_0}{V}\right) \dot{V} + a_3 \dot{L} + a_4 \dot{B} \\ &= F_1(S, V) + F_2, \end{aligned} \tag{8}$$

where

$$\begin{aligned} F_1(S, V) &= a_1(S - S_0) \left[q_1 \mu \left(\frac{1}{S} - \frac{1}{S_0} \right) + \varepsilon \left(\frac{V}{S} - \frac{V_0}{S_0} \right) \right] \\ &\quad + a_2(V - V_0) \left[q_2 \mu \left(\frac{1}{V} - \frac{1}{V_0} \right) + p \left(\frac{S}{V} - \frac{S_0}{V_0} \right) \right], \end{aligned} \tag{9}$$

$$\begin{aligned} F_2 &= -a_1 \beta(S - S_0)(L + B) + a_2 \left(1 - \frac{V_0}{V}\right) (\gamma_1 L + \gamma_2 B) \\ &\quad - a_2 \delta \beta(V - V_0)(L + B) + a_3 \beta(L + B)(S + \delta V) - a_3(\gamma_1 + \alpha + \mu)L \\ &\quad + a_4 \alpha L - a_4(\gamma_2 + \mu)B. \end{aligned} \tag{10}$$

Let $a_1 = a_2 = a_3$, and it is also easy to verify there exist $a_4 > 0$ that

$$F_2 \leq \frac{(\gamma_1 + \alpha + \mu)(\gamma_2 + \mu)}{(\gamma_2 + \mu + \alpha)} (R_0 - 1)(L + B). \tag{11}$$

Thus, we can get

$$\dot{G} \leq \frac{(\gamma_1 + \alpha + \mu)(\gamma_2 + \mu)}{(\gamma_2 + \mu + \alpha)} (R_0 - 1)(L + B) + F_1(S, V). \tag{12}$$

For $F_1(S, V)$, let $x = \frac{S}{S_0}, y = \frac{V}{V_0}$ then

$$\begin{aligned} F_1(S, V) &= a_1 \left[q_1 \mu (x - 1) \left(\frac{1}{x} - 1 \right) + \varepsilon V_0 (x - 1) \left(\frac{y}{x} - 1 \right) + q_2 \mu (y - 1) \left(\frac{1}{y} - 1 \right) \right. \\ &\quad \left. + p S_0 (y - 1) \left(\frac{y}{x} - 1 \right) \right] \\ &= a_1 \bar{F}(x, y). \end{aligned} \tag{13}$$

From Eq. (5), one can derive:

$$\begin{aligned} \bar{F}(x, y) &= (2\mu q_1 + 2\mu q_2 + \varepsilon V_0 + p S_0) - \mu S_0 x - \mu q_1 \frac{1}{x} - \mu V_0 y - \mu q_2 \frac{1}{y} - \varepsilon V_0 \frac{y}{x} \\ &\quad - p S_0 \frac{x}{y}. \end{aligned} \tag{14}$$

To show that $\bar{F}(x, y) \leq 0$, we rearrange the terms in $\bar{F}(x, y)$ by the following form: $H := \sum_{k=1}^m b_k (n_k - h_{k,1} - h_{k,2} - \dots - h_{k,n_k})$, where $b_k \geq 0$, $h_{k,i}, i = 1, 2, \dots, n_k$ is a monomial of x, y and $\prod_{i=1}^{n_k} h_{k,i} = 1$. According to the property that the arithmetic mean is greater than or equal to the geometric mean, i.e.

$h_{k,1} + h_{k,2} + \dots + h_{k,n_k} - n_k \geq 0$, one can derive is negative semidefinite. Note that, for the nonconstant term of $\bar{F}(x, y)$, the groups satisfy $\prod_{i=1}^{n_k} h_{k,i} = 1$ totally have the following five cases:

$$\left\{x, \frac{1}{x}\right\}, \left\{y, \frac{1}{y}\right\}, \left\{\frac{y}{x}, \frac{x}{y}\right\}, \left\{x, \frac{1}{y}, \frac{y}{x}\right\}, \left\{y, \frac{1}{x}, \frac{x}{y}\right\}.$$

Then, corresponding to Eq. (14), we have

$$H(x, y) = b_1\left(2 - x - \frac{1}{x}\right) + b_2\left(2 - y - \frac{1}{y}\right) + b_3\left(2 - \frac{y}{x} - \frac{x}{y}\right) + b_4\left(3 - x - \frac{1}{y} - \frac{y}{x}\right) + b_5\left(3 - y - \frac{1}{x} - \frac{x}{y}\right). \tag{15}$$

Let $\bar{F}(x, y) = H(x, y)$ to determine the coefficients $b_i (i = 1, 2, \dots, 5)$, which gives

$$\begin{cases} b_1 + b_4 = \mu S_0, \\ b_1 + b_5 = \mu q_1, \\ b_2 + b_5 = \mu V_0, \\ b_2 + b_4 = \mu q_2, \\ b_3 + b_4 = \varepsilon V_0, \\ b_3 + b_5 = pS_0. \end{cases} \Rightarrow \begin{cases} b_1 = \mu q_1 - b_5, \\ b_2 = \mu V_0 - b_5, \\ b_3 = pS_0 - b_5, \\ b_4 = \varepsilon V_0 - pS_0 + b_5. \end{cases}$$

In order to ensure $b_1, b_2, b_3, b_4 \geq 0$, b_5 should satisfy $pS_0 - \varepsilon V_0 \leq b_5 \leq \min\{\mu q_1, \mu V_0, pS_0\}$. According to the value of V_0, S_0 , the inequality $pS_0 - \varepsilon V_0 \leq b_5 \leq \min\{\mu q_1, \mu V_0, pS_0\}$ exists permanently. From the relationship among $b_i, i = 1, 2, \dots, 5$, we can derive that the constant term of $\bar{F}(x, y)$ and $H(x, y)$ is equal. According to the property that the arithmetic mean is greater than or equal to the geometric mean, we derive $H(x, y) \leq 0$, that is $\bar{F}(x, y) \leq 0$. Then,

$$\dot{G} \leq \frac{(\gamma_1 + \alpha + \mu)(\gamma_2 + \mu)}{(\gamma_2 + \mu + \alpha)} (R_0 - 1)(L + B). \tag{16}$$

Thus, when $R_0 \leq 1$, we have $\dot{G} \leq 0$. Due to the virus-free equilibrium, P_0 is the unique invariant set of system (1) on the set Ω . Therefore, by the LaSalles Invariance Principle, when $R_0 \leq 1$, virus-free equilibrium P is globally stable in the feasible region Ω . □

Theorem 1 tell us that, in order to contain viruses, it is vital to take preemptive measures so that R_0 is well-below one. Figure 2 displays the time plot of S, V, L, B . It follows by Theorem 1 that the virus equilibrium is globally asymptotically stable with $R_0 < 1$.

From Fig. 3, one may observe the low disconnected (connected) rate μ has a relatively high basic reproduction number R_0 and the disappear time of computer virus. With the increase of μ , the value of R_0 becomes small, but the decay rate is slow. The disappear time of computer virus for $\mu = 0.1$ is about 4.5 times for $\mu = 1$.

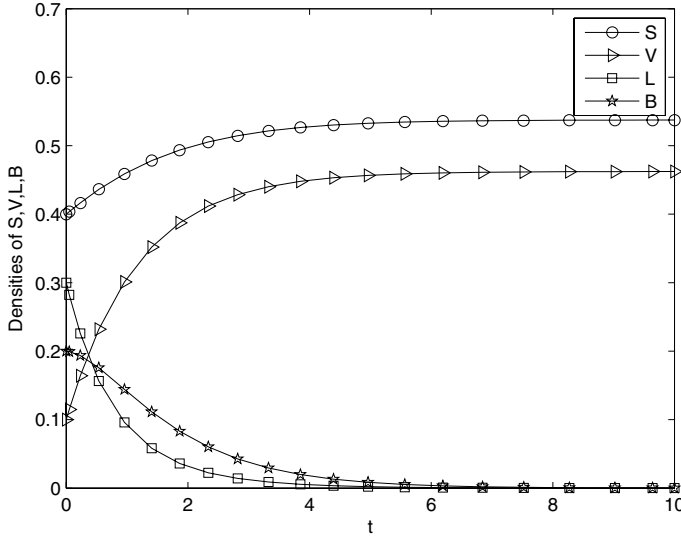


Fig. 2. Consider Eq. (1) with $q_1 = 0.79, q_2 = 0.21, \beta = 0.2, \varepsilon = 0.1, r_1 = 0.1, r_2 = 0.23, \delta = 0.1, \alpha = 0.59, \mu = 0.69, p = 0.41$. By formula, we have $R_0 = 0.15$.

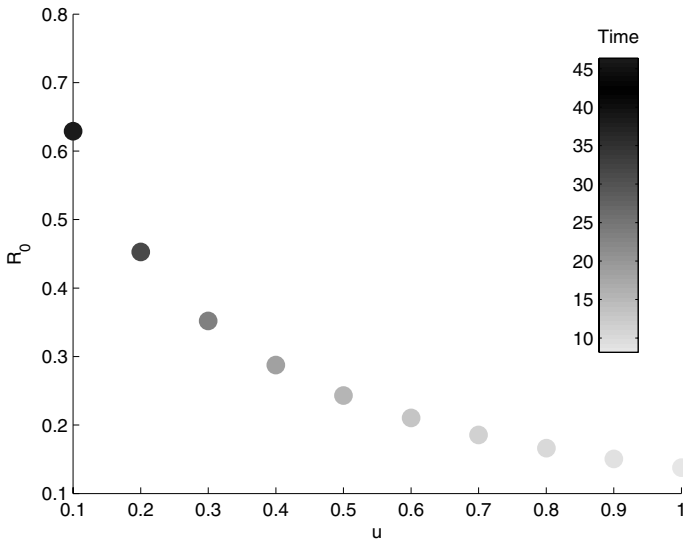


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4.2. The viral equilibrium locally stable

Theorem 2. When $\eta_0 > 0, \eta_1 > 0, \eta_2 > 0, \eta_3 > 0, \eta_4 > 0, \eta_1\eta_2 - \eta_0\eta_3 > 0, \eta_3(\eta_1\eta_2 - \eta_0\eta_3) - \eta_1^2\eta_4 > 0$, the viral equilibrium P^* is locally asymptotically stable. Where $\eta_0, \eta_1, \eta_2, \eta_3, \eta_4$ is determined by Jacobian matrix and Routh stability criterion.

Table 1. Routh array of characteristic equation.

Identifier number	First column	Second column	Third column	Fourth column
λ^4	η_0	η_2	η_4	0
λ^3	η_1	η_3	0	0
λ^2	b_1	η_4	0	0
λ^1	c_1	0	0	0
λ^0	η_4	0	0	0

Proof. The Jacobian matrix of Eq. (1) at the viral equilibrium can be written as

$$J(P^*) = \begin{pmatrix} -\beta(L^* + B^*) - (\mu + p) & \varepsilon & -\beta S^* & -\beta S^* \\ p & -\delta\beta(L^* + B^*) - (\mu + \varepsilon) & \gamma_1 - \beta\delta V^* & \gamma_2 - \beta\delta V^* \\ \beta(L^* + B^*) & \delta\beta(L^* + B^*) & \beta(S^* + \delta V^*) - (\gamma_1 + \alpha + \mu) & \beta(S^* + \delta V^*) \\ 0 & 0 & \alpha & -(\gamma_2 + \mu) \end{pmatrix},$$

where we denote the characteristic equation of matrix $J(P^*)$ with

$$|J - \lambda E| = \eta_0\lambda^4 + \eta_1\lambda^3 + \eta_2\lambda^2 + \eta_3\lambda + \eta_4 = 0, \tag{17}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are eigenvalues of the matrix, which are determined by $|J - \lambda E| = 0$.

The Routh array of the system is described in Table 1.

In Table 1, where $b_1 = \frac{\eta_1\eta_2 - \eta_0\eta_3}{\eta_1}$ and $c_1 = \eta_3 - \frac{\eta_1^2\eta_4}{\eta_1\eta_2 - \eta_0\eta_3}$. According to the Routh stability criterion, system (1) behaves localized asymptotic stability at P^* , if $\eta_0 > 0$, $\eta_1 > 0, \eta_2 > 0, \eta_3 > 0, \eta_4 > 0, \eta_1\eta_2 - \eta_0\eta_3 \geq 0, \eta_3(\eta_1\eta_2 - \eta_0\eta_3) - \eta_1^2\eta_4 > 0$. \square

5. Simulation and Discussion

In this section, we study how the different parameters influences computer virus spreading.

It can be seen from Fig. 4, an increase in disconnecting rate from the Internet could lead to higher speed of the eradication of computer virus, that is, the higher disconnecting rate is beneficial to control the spread of computer virus. Hence, it is highly recommended that one disconnects his computer from the Internet, whenever this connection is unnecessary.

It is shown that in Fig. 5, the higher proportion of installing with anti-virus software is beneficial to control the spread of computer virus. When the proportion is extremely small, such as $p = 0.1$, the computer virus would not be eradicated. This informs us that installing anti-virus software is an important way to control the spread of virus. Therefore, it is strongly recommended that one should periodically acquire and run anti-virus software of the newest version.

To illustrate the influence of μ and p on computer virus spreading in detail, we calculate the final density $L(\infty) + B(\infty)$. Figure 6 shows how the final proportion of $L(\infty) + B(\infty)$ changes with μ and p . It shows that, with the increase of μ and p , the proportion of $L(\infty) + B(\infty)$ is decrease. We see that $L(\infty) + B(\infty)$ has larger values

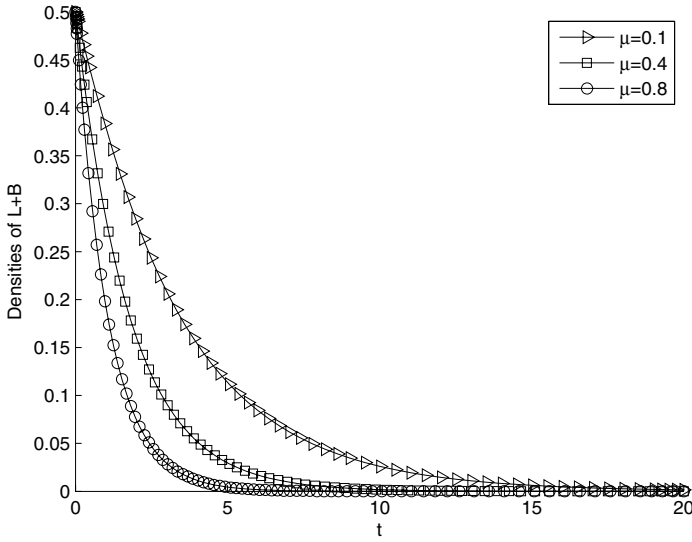


Fig. 4. Consider Eq. (1) with $q_1 = 0.69, q_2 = 0.31, \beta = 0.25, \varepsilon = 0.12, r_1 = 0.22, r_2 = 0.33, \delta = 0.21, \alpha = 0.5, p = 0.31$. Evolution of $L + B$ for different values of μ .

when μ is small and then gradually decreases to a small value when μ is relatively large. More interesting is that when μ is close to 1 but p is close to 0, the proportion of infected computers are zero. This result can be qualitatively explained as follows: that an anti-virus software is updated always after the computer is connected to the

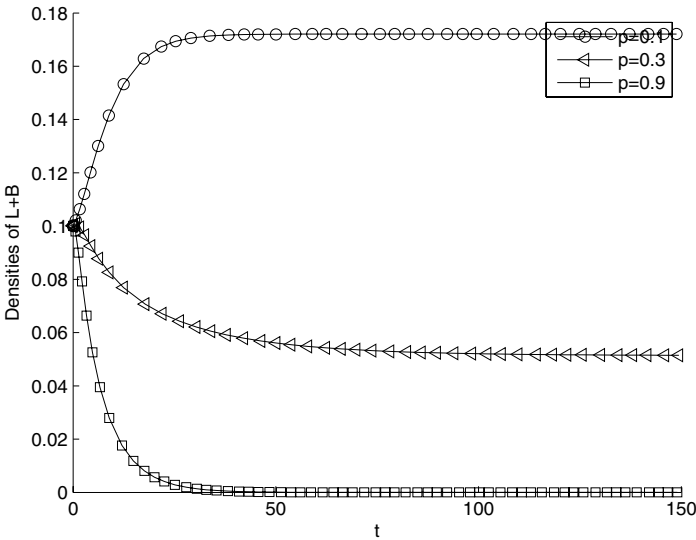


Fig. 5. Consider Eq. (1) with $q_1 = 0.89, q_2 = 0.11, \beta = 0.7, \varepsilon = 0.5, r_1 = 0.2, r_2 = 0.33, \delta = 0.1, \alpha = 0.5, \mu = 0.2$. Evolution of $L + B$ for different values of P .

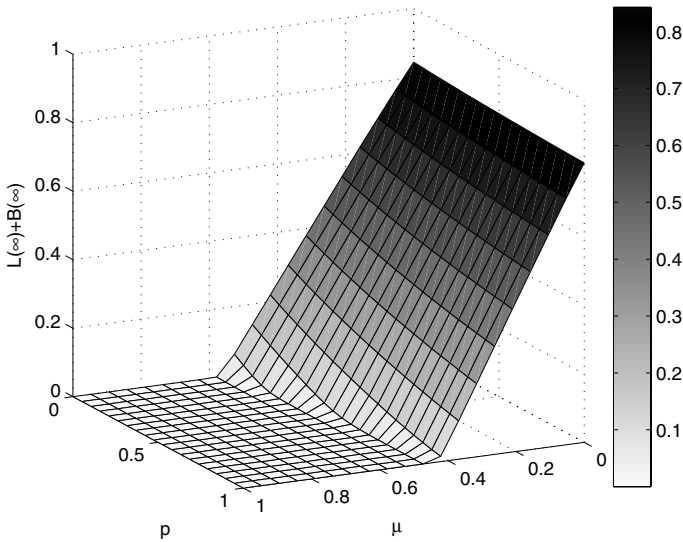


Fig. 6. Consider Eq. (1) with $q_1 = 0.6, q_2 = 0.4, \beta = 0.7, \varepsilon = 0.5, r_1 = 0.05, r_2 = 0.1, \delta = 0.5, \alpha = 0.45$. The effect of μ, p on the final proportion of $L(\infty) + B(\infty)$.

Internet. Therefore, it is more important that one disconnects his computer whenever this connection is unnecessary than periodically run and update the anti-virus software.

6. Conclusions

In this paper, we have focused on the impact of anti-virus software on the spreading of computer virus by developing a mathematical model. Considering that the anti-virus software may not be too much effective as it may be an outdated version or it may not be updated, that is, although a computer with anti-virus software, they can still re-acquire the virus with a reduced incidence rate $\delta\beta$, where $0 < \delta < 1$. Besides, we consider two different kinds of infected computers (latent computers and breaking computers). By introducing appropriate Lyapunov function and the Routh stability criterion, acquiring the stability conditions of the virus-free equilibrium and virus equilibrium. Finally, we give some advices to control the spread of a computer virus on the Internet.

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