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Reorganizations of complex networks: Compounding and reducing

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Real networks interact with each other by different kinds of topological connections, which are usually demonstrated by linking nodes of different networks. Simple connection, such as one-to-one corresponding, random connection and similar connection are adopted for studying the interacted networks. Practical interrelations established between the two networks are ignored. In this study, a generalized framework of multi-subnet composited complex network that allowed us to investigate interrelations among several subnets is developed. Based on that, reorganizations of networks: compounding (compound subnets into a "bigger" one) and reducing (obtain a "smaller" network from a "bigger" one) are proposed. As an empirical evidence, influence of compounding on traffic dynamics is discussed. And the properties of nodes linking two networks are also considered. Onset of compounding between two networks is revealed. Numerical simulations on artificial networks as well as real bus and tube networks of Qingdao in China agree well with our analysis, which show validity of our model.

Keywords: Complex network; reorganizations of networks; traffic dynamic.

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1. Introduction

Complex system generally consists of many kinds of parts and interrelations. For simplicity, one kind of parts and their one kind of interrelations are described by complex network. However, analysis involving several kinds of parts and interrelations of complex system is rather common. For example, traffic system contains travelers, vehicles and roads, in which they interact with each other in different ways. Some problems, such as transfer, could not be studied by isolated road network, bus network and tube network since it is not up to any of them entirely. Similarly, cascading failure on infrastructures system (such as communication and power grid networks[1,2]), identifying groups in multi-mode networks (such as three-mode

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network of YouTube, users, videos and tags[3,4]), also involve several kinds of parts and interrelations of the system. With those networks describing one kind of parts and their one kind of interrelations of complex systems, when studying problem involving several kinds of parts and interrelations, one feasible way is to compound them into a "bigger" one and those composited networks could be further composited into a new one again. For network involving several kinds of parts and interrelations, reducing it into a "smaller" one in which only those parts and interrelations which are needed are reserved is also required. That is to say, reorganizations of complex networks, such as compounding and reducing, according to practical issues could help analyze the structure and dynamics of network in a better way.

In the view of graph, reorganizations of complex networks relate with graph operations. Graph operations means creating a new graph from initial graphs, such as disjoint union, graph join, product of graphs (Cartesian product,[5] Kronecker product,[6] Lexicographic product[7]). However, these efforts do not distinguish the kind of nodes (or links) and regarded them as homogenous ones.

Recently, layered network was presented by Kurant and Thiran[8] and Kurant,[9] where nodes and links in one layer represent one kind of parts and their one kind of interrelations. With regard to two layers or networks connecting, interacting networks,[10] interdependent networks,[11] interconnecting networks[12] and coupled networks[13] have been proposed. Gu *et al.*[14] proposed that there is one-to-one correspondence between nodes of two layers. They studied traffic dynamic between the two layer networks. Cases of randomly connected nodes in different layers are studied by cascading failure.[15–19] Parshani *et al.*[20] found that real nodes in two networks are usually not randomly connected, rather are coupled according to some regularity which they called inter-similarity. Cho *et al.*[21] explored the impact of kinds of nodes degree distribution on the properties of a giant component of the inter-network similarity. In the above literature, simple connection, such as one-to-one correspondence, random connection or similar connection, are adopted. However, interrelation we want the two networks would establish may play more important role. For example, a metro network is planned to connect a bus network, topologies of two connection, one with aim of minimizing the traveling time of passengers, another one with aim of minimizing construction cost, may be greatly different. That is to say, overall functionalities of the two networks are influenced a lot by the interrelation which would be established between the two networks.

In this study, we propose a generalized complex network model of describing complex system involving several kinds of parts and interrelations. Through bringing the interrelations between parts into formalized definition of complex network, a four-tuple named multi-subnet composited complex network is given. We show that complex network describing one kind of parts and their one kind of interrelations is a special case in our model. Based on that, we give formalized definitions of connecting complex networks (named compounding) in term of interrelations of nodes. Compounding is to composite two networks into a bigger one. Another operation of complex network reducing has been proposed to obtain a smaller network from the bigger one. Based on the model, transfer between bus and tube networks is studied by analyzing traffic dynamic on composited network. Three kinds of connection styles are discussed. And the properties of different kinds of nodes are also considered. Onset of compounding between two networks is revealed. Numerical simulations on artificial networks as well as real bus and tube networks of Qingdao in China agree well with our analysis, which show validity of our model.

2. Model of Dynamic Organizations of Complex Networks

Usually complex networks are described as $G = (V, E)$, where V is a finite set of nodes and E is set of links of nodes. Interrelations among these nodes are omitted. In this study, we regard that networks could be compounded by establishing new interrelations among nodes, whereas network also could be reduced by wiping off interrelations among nodes. In this way, four-tuple description of complex network is given as follows.

Definition 1 (Multi-subnet Composited Complex Network). A multi-subnet composited complex network is four-tuple $G = (V, E, R, F)$, where

(1) $V = \{\upsilon_1, \upsilon_2, \ldots, \upsilon_m\}$ is a finite set of nodes and $m = |V|$;
(2) $E = \{\langle \upsilon_h, \upsilon_l\rangle | \upsilon_h, \upsilon_l \in V, 1 \le h, l \le m\} \subseteq V \times V$ is a finite set of links between nodes;
(3) $R = R_1 \times \cdots \times R_i \times \cdots \times R_n = \{(r_1, \ldots, r_i, \ldots, r_n) \,|\, r_i \in R_i, 1 \le i \le n\}$, where R_i is one kind of interrelations between nodes and the amount of kinds of interrelations is n;
(4) F is a mapping from E to R.

Mapping F gives each link a n-tuple to denote all kinds of interrelations of its two nodes. Let $r_i = \phi$ demonstrate that there is no interrelation R_i between the nodes.

Due to complexity of complex system, we usually model one aspect of the system as subnet. Whereas for some problems, analysis referring several aspects of the system is essential. In such cases, we expect that several subnets could be composited together. By considering that, we define subnet as follows.

Definition 2 (Subnet). Assuming composited network $G = (V, E, R, F)$, $G' = (V', E', R', F')$ is said to be subset of G according to set of interrelations $R'(R' \subseteq R$ and $R' \neq \Phi)$, if and only if

(1) $V' \subseteq V$;
(2) $E = \{\langle \upsilon_h, \upsilon_l\rangle | F(\langle \upsilon_h, \upsilon_l\rangle) \in R', \langle \upsilon_h, \upsilon_l\rangle \in E, \upsilon_h, \upsilon_l \in V\}$;
(3) $F' : E' \to R'$. For $\forall\langle \upsilon'_h, \upsilon'_l\rangle \in V'$, there is $F'(\langle \upsilon'_h, \upsilon'_l\rangle) = F(\langle \upsilon'_h, \upsilon'_l\rangle)$.

Definition 3 (Two-tuples of Compounding Mapping). Given two subnets $G_1 = (V_1, E_1, R_1, F_1)$, $G_2 = (V_2, E_2, R_2, F_2)$, $V'_1 \subseteq V_1$, $V'_2 \subseteq V_2$, R' is set of interrelations, $r' \in R'$, (Ξ, Ψ) is called two-tuples of compounding mapping, where $\Xi : V'_1 \to V'_2$ and $\Psi : \{\langle \upsilon, \Xi(\upsilon)\rangle | \upsilon \in V'_1, \Xi(\upsilon) \in V'_2\} \to \{r'\}$.

Nodes and interrelations which would be established among them are given in two-tuples of compounding mapping. $\langle \upsilon, \Xi(\upsilon)\rangle$ is called outer links and υ, $\Xi(\upsilon)$ are called marginal nodes. r' is called compounding interrelation.

Definition 4 (Subnet Compounding). Given two subnets $G_1 = (V_1, E_1, R_1, F_1)$, $G_2 = (V_2, E_2, R_2, F_2)$, where $R_1 = R_{11} \times \cdots \times R_{1i} \times \cdots \times R_{1n_1} = \{(r_{11}, \ldots, r_{1i}, \ldots, r_{1n_1})|r_{1i} \in R_{1i}, 1 \leq i \leq n_1\}$, $R_2 = R_{21} \times \cdots \times R_{2j} \times \cdots \times R_{2n_2} = \{(r_{21}, \ldots, r_{2j}, \ldots, r_{2n_2})|r_{2j} \in R_{2j}, 1 \leq j \leq n_2\}$, two-tuples of compounding mapping (Ξ, Ψ), where r' is compounding interrelation, compounding subnet G_1 to G_2 means generating a new composited network $G = (V, E, R, F)$, where

(1) $V = V_1 \cup V_2$;
(2) $E \subseteq E_1 \cup E_2 \cup D_\Psi$, where D_Ψ is domain of definition of Ψ;
(3) $R = \{(r_1, \ldots, r_{k'}, \ldots, r', \ldots, r_{k''}, \ldots, r_n)|r_{k'} \in R_{1k'}, r_{k''} \in R_{2k''}, r_{k'} \neq r_{k''}, 1 \leq k' \leq n_1, 1 \leq k'' \leq n_2, 1 \leq n \leq n_1 + n_2 + 1\}$;
(4) $F : E \to R$, when $\langle \upsilon_h, \upsilon_l\rangle \in E_1$, $F(\langle \upsilon_h, \upsilon_l\rangle) = F_1(\langle \upsilon_h, \upsilon_l\rangle), \phi, \ldots, \phi)$; when $\langle \upsilon_h, \upsilon_l\rangle \in E_2$, $F(\langle \upsilon_h, \upsilon_l\rangle) = (\phi, \ldots, \phi, F_2(\langle \upsilon_h, \upsilon_l\rangle))$; when $\langle \upsilon_h, \upsilon_l\rangle \in V_1 \times V_2$, $F(\langle \upsilon_h, \upsilon_l\rangle) = (\phi, \ldots, \phi, r', \phi, \ldots, \phi)$, where $\langle \upsilon_h, \upsilon_l\rangle \in V$, $1 \leq h$, $l \leq |V|$.

The new composited network is decided by two-tuple of compounding mapping. Even with the same compounding interrelation the outcome of subnet compounding may be different.

Definition 5 (Subnet Reducing). Given composited network $G = (V, E, R, F)$, set of interrelations $R' \subset R$, subnet reducing means obtaining subnet $G' = (V', E', R', F')$ of G according to set of interrelations R', where

(1) $V' \subseteq V$;
(2) $E' = \{\langle \upsilon_h, \upsilon_l\rangle | F(\langle \upsilon_h, \upsilon_l\rangle) \in R', \langle \upsilon_h, \upsilon_l\rangle \in E, \upsilon_h, \upsilon_l \in V\}$, $1 \leq h, l \leq |V|$;
(3) $R' \subset R$;
(4) Mapping $F' : E' \to R'$, $F'(\langle \upsilon_h, \upsilon_l\rangle) = F(\langle \upsilon_h, \upsilon_l\rangle)$;

R' is called as reduced interrelations. In contrast with subnet compounding, subnet reducing depends on reduced interrelations.

3. Application of Subnet Compounding

In this section, we show that some problems could be solved effectively by the proposed organization of complex networks. As mentioned above, transfer in public transportation involve bus network and tube network. In the following, we would show how to reduce travel time of passengers on the two networks by compound tube network and bus network.

3.1. *Traffic dynamic model of composited network*

Supposing $G_1 = (V_1, E_1, R_1, F_1)$, $G_2 = (V_2, E_2, R_2, F_2)$, where G_1, G_2 represents bus network and tube network, respectively. V_1, V_2 are sets of stations. Nearby stations on one bus (or tube) route join as links forming sets of E_1, E_2. $r_1 \in R_1$, $r_2 \in R_2$ mean relation of one bus (or tube) route through by between two stations. Mapping F_1, F_2 describe whether there is route passing through between two nearby stations.

Given compounding interrelation r, r means that passengers could transfer from bus network to tube network and vice versa. By compounding G_2 and G_1, passengers may change their lines, such as travel to G_2 and then back to G_1. It is up to the regulation of which line taking least cost. Set time cost of passing one link of G_1 is 1 and that of G_2 is q and that of linking nodes of G_1 and G_2 is p, as shown in Fig. 1.

Given two nodes $x, y \in V_1$ in G_1 as starting point and ending point, there are two kinds of ways passengers taking: one is traveling only in G_1 (only refers to interrelation r_1); another one is traveling in both G_1 and G_2 (refers to interrelations r_1, r_2 and r). Let d_{xy}^z indicate the length of shortest path between $x, y \in V_1$ when traveling by interrelations in Z. Let c_1, c_2 be the time cost of traveling in G_1 and both in G_1 and G_2, respectively. Then, $c_1 = d_{ij}^{r_1} \times 1$ and $c_2 = 1 \times d_{iu_1}^{r_1} + p \times d_{u_1u_2}^{r} + q \times d_{u_2v_2}^{r_2} + p \times d_{v_2v_1}^{r} + 1 \times d_{v_1j}^{r_1}$. When $c_1 < c_2$, passenger would choose to travel only in G_1; whereas $c_1 > c_2$ passengers would choose to travel both in G_1 and G_2. Then the transition point could be obtained, when $c_1 = c_2$ and $d_{u_1u_2}^{r} = d_{v_2v_1}^{r} = 0$. That is $d_{ij}^{r_1} = (d_{iu_1}^{r_1} + d_{v_1j}^{r_1}) + q \times d_{u_2v_2}^{r_2}$. Considering that $d_{ij}^{r_1} - (d_{iu_1}^{r_1r_2r_3} + d_{v,j}^{r_1r_2r_3}) \approx d_{u_1v_1}^{r_1}$, the above equation could be rewrote as $d_{u_1v_1}^{r_1} = q \times d_{u_2v_2}^{r_2}$. Threshold value of q is $q_c = \frac{d_{u_1v_1}^{r_1}}{d_{u_2,v_2}^{r_2}}$. Obviously, compounding mapping (which nodes in V_1, V_2 being chosen as marginal nodes) influence the value of q_c. We suggest three simple compounding mappings:

(1) Random mapping. Select nodes randomly in two networks. Though not being feasible in practice, this way counts as reference to the other two methods.

(2) "Weak–strong" mapping. Without considering spatial constraint, for nodes which are far away from others in one network, one effective way to reduce distance to others is to link it to one node which locates in center. We define closeness of node as $tc_{v_h} = \frac{\min_{v_g \in V} \sum_{v_l \in V} d_{v_g v_l}}{\sum_{v_l \in V} d_{v_h v_l}}$, where $d_{v_h v_l}$ represents the distance

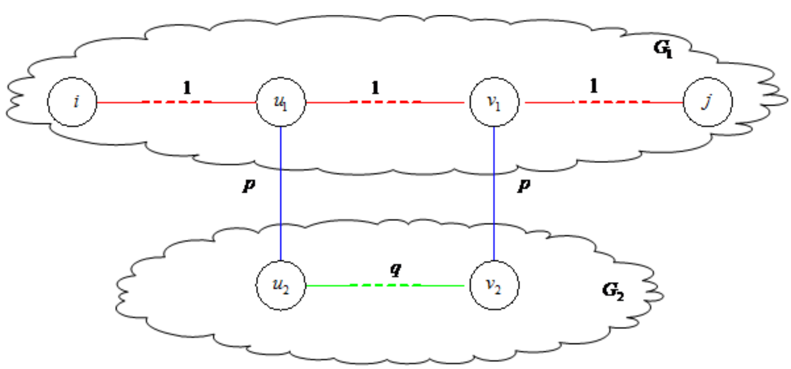


Fig. 1. (Color online) Illustration of compounding G_1 and G_2. Red links represent that there are bus route passing through, blue links represent transfer channel between bus and tube and green links represent tube routes passing through.

between node $\upsilon_h, \upsilon_l \in V, 1 \leq g, h, l \leq |V|$. When $tc_{\upsilon_h} = 1$ means that υ_h is center of the network, whereas more closer to zero means that υ_h is far away from other nodes. We preferentially select nodes with smaller closeness (weak nodes) in G_1 and nodes with larger closeness (strong nodes) in G_2.

(3) "Matched" mapping. Parshani *et al.*[20] founded that real interdependent networks are usually not randomly interdependent, rather a pair of dependent nodes are coupled according to some regularity which they coined as inter-similarity. Inspired by that, the third way is to select nodes with similar closeness in G_1 and G_2. Given threshold ρ, randomly select nodes $\upsilon_g \in V_1$ and $\upsilon_l \in V_2$, where $|tc_{\upsilon_g} - tc_{\upsilon_h}| \leq \rho$.

Let L^{r_1} and L^{r_2} demonstrate the average length of shortest paths of G_1 and G_2, respectively. Theoretical analysis of q_c is given as follows:

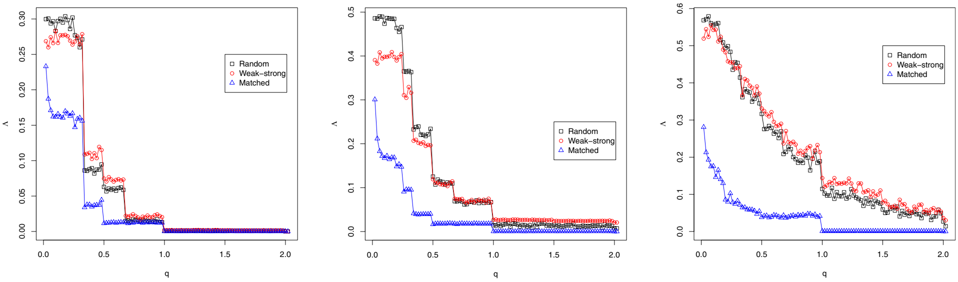
(1) Topology of G_1 and G_2 are the same ($L^{r_1} = L^{r_2}$). When selecting marginal nodes randomly, then $d^{r_1}_{u_1\upsilon_1} \approx L^{r_1}$ and $d^{r_2}_{u_2\upsilon_2} \approx L^{r_2}$. Substituting them into q_c, then $q_c \approx 1$. When selecting marginal nodes in "Weak–strong" way, $d^{r_1}_{u_1\upsilon_1} \geq L^{r_1}$ and $d^{r_2}_{u_2\upsilon_2} \leq L^{r_2}$. Then $q_c \geq 1$. When selecting marginal nodes in "Matched" way, $d^{r_1}_{u_1\upsilon_1} \approx d^{r_2}_{u_2\upsilon_2}$, then $q_c \approx 1$.

(2) Topology of G_1 and G_2 are different (assuming $L^{r_1} > L^{r_2}$). When selecting marginal nodes randomly, then $d^{r_1}_{u_1\upsilon_1} \approx L^{r_1}$ and $d^{r_2}_{u_2\upsilon_2} \approx L^{r_2}$. Substituting them into q_c, then $q_c > 1$. When selecting marginal nodes in "Weak–strong" way, $d^{r_1}_{u_1\upsilon_1} \geq L^{r_1}$ and $d^{r_2}_{u_2\upsilon_2} \leq L^{r_2}$. Then, $q_c > 1$. When selecting marginal nodes in "Matched" way, then $d^{r_1}_{u_1\upsilon_1} \approx L^{r_1}$ and $d^{r_2}_{u_2\upsilon_2} \approx L^{r_2}$, then $q_c \approx 1$.

3.2. *Numerical simulations*

At first, we simply assume that there is one passenger traveling between each pair of nodes on G_1. Set parameters Λ representing the number of passengers who change their original lines in G_1 and travel between G_1 and G_2. Figure 2 shows results of numerical simulation with $|V_1| = |V_2| = 100$, mean degree of nodes $\langle k \rangle = 6$ and $\rho = 0.05$. The number of outer links is 50. Each curve on these charts is average result of one hundred times. As shown in Fig. 2, we find that regardless of topology Λ changes from positive to zero (except WS network with random and "weak–strong" compounding mapping) at transition point $q = 1$. That shows passengers would not choose tube network when traveling one link of tube network takes more time than that of bus network. Another point should be valued is, when choosing marginal nodes in "matched" compounding mapping the number of passengers who choose switching between the two networks is least. It seems like the third way of "matched" compounding mapping induces less passengers taking tube for transferring.

Second, we assume that passengers are added with a given rate μ (passengers per time-step) at each node of G_1 and each passenger is assigned a random destination. At each step, each node can deliver passengers toward their destinations and first-in-first-out (FIFO) queuing discipline is applied at each node. Assuming that the length



(a) (b) (c)

Fig. 2. (Color online) Λ as a function of q, where topology of G_1 is same with G_2 (a) BA networks; (b) ER networks (the connecting probability is 0.06); (c) WS networks (the connecting probability is 0.01 and the number of nearest neighbors is three).

of queen is infinite. Once a passenger arrives at destination, it will be removed from the network. Let the ratio of number of delivered passengers by marginal nodes to that by nonmarginal nodes at each time-step be parameter s.

We simulate with $V_1 = V_2 = 100$, $\langle k \rangle = 6$ and $s = 1$. The number of outer links is 50 and the overall time-step is 500. Figure 3 shows the average traveling time $\langle T \rangle$ of passengers as function of q. The results illustrate us that average traveling time of passengers cost least, when choosing marginal nodes in "matched" compounding mapping. That could be explained that due to many passengers arriving at marginal nodes for saving travel time and then they need to wait for going on. It is worse

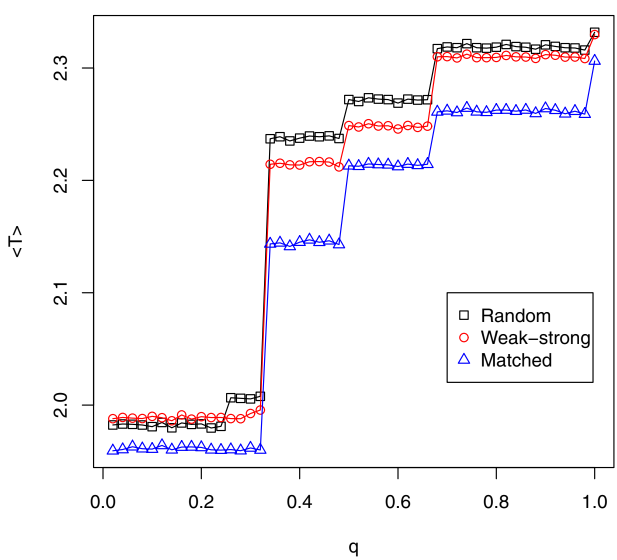


Fig. 3. (Color online) $\langle T \rangle$ as function of q. G_1, G_2 are BA networks.

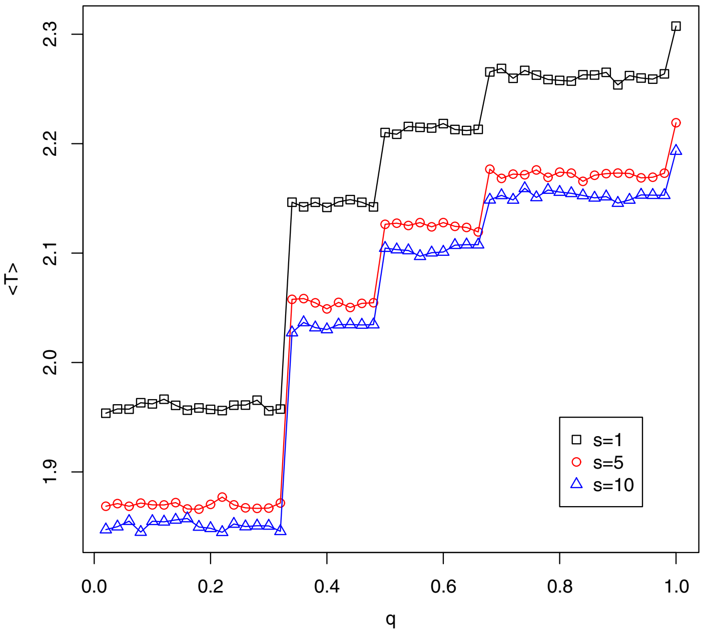


Fig. 4. (Color online) Influence of s on $\langle T \rangle$ as a function of q, G_1, G_2 are BA networks.

when marginal nodes are chosen in "weak–strong" compounding mapping as depicted in Fig. 2.

For showing the explanations further, influence of s on $\langle T \rangle$ is analyzed in Fig. 4. With the increment of s, $\langle T \rangle$ reduces showing that strengthening the capability of transferring of marginal nodes could reduce average travel time of passengers.

3.3. *Validations of empirical data*

Validations of the above traffic model by empirical bus and tube data of Qingdao in China is given. Bus data are collected from www.8684.com (updating until 7 July 2012). Bus routes in satellite town are not included. Route with different up-link and bottom-link are omitted. Suppose that the station name is an identifier of each station. The nodes in bus network are represented by bus stops and edges are generated when there exists more than one bus route between nearby stops, that is, relation between nodes is bus route passing through by. Then we can obtain bus network $G_{\text{bus}} = (V_{\text{bus}}, E_{\text{bus}}, r_{\text{bus}}, F_{\text{bus}})$, where $V_{\text{bus}} = 875$ and $E_{\text{bus}} = 1420$. Mapping F_{bus} indicates whether relation r_{bus} exists between nodes. For tube routes, since they have not been established yet, two routes M3 and M2 which would be finished are collected from www.ccmetro.com/newsite/readnews.aspx?id = 65025. Nodes, edges, relation between nodes and mapping from edges to relationships are defined as those in bus network. Then we could get $G_{\text{metro}} = (V_{\text{metro}}, E_{\text{metro}}, r_{\text{metro}}, F_{\text{metro}})$, where $V_{\text{metro}} = 55$, $E_{\text{metro}} = 55$. Actually, stations with the same names could be regarded

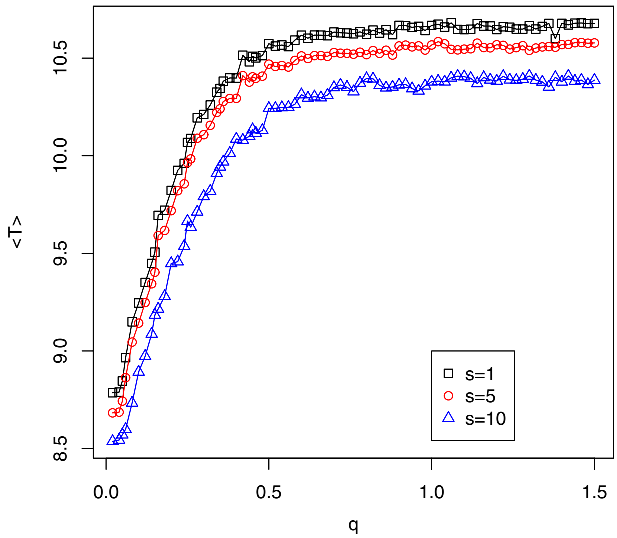


Fig. 5. (Color online) Influence of s on $\langle T \rangle$ as a function of q with G_{bus}, G_{metro}.

as transfer stops for passengers switching between bus and tube. Therefore, we let the nodes with same names as marginal nodes and then link them according with their names as outer links. Then, the two networks are composited by subnet compounding. Let the travel time of each edge passing by passengers is 1 in G_{bus} and that in G_{metro} is q and that of the outer links between these two networks is $p = \frac{1+q}{2}$.

We discuss the influence of s on average traveling time $\langle T \rangle$. As shown in Fig. 5, when we increase s, $\langle T \rangle$ reduces. Notice that, in the real case, the two networks contain widely different number of nodes, and topologies of them could not obey BA, ER or WS well. Therefore, the results cannot be quantitatively predicted by the simulations. However, compared with the results in Fig. 4, the empirical results agree with the simulations qualitatively, that is, the strength of marginal nodes could reduce average travel time of passengers and with increase of q $\langle T \rangle$ reduces. Another point could be used to verify our model, is that, when q reached to value 1 $\langle T \rangle$ would be around 10.5 more or less, as shown in Fig. 5. Actually, when q reached to value 1, that means the cost of traveling each link in G_{bus} and G_{metro} is the same. And then passengers would not choose to transfer between bus and tube; that is to say, $\langle T \rangle$ would be the cost of traveling only in G_{bus}, which must be equal to the average length of G_{bus}. The average length of G_{bus} is 10.5.

4. Conclusions

Recently, complex network as one kind model of complex system has been aroused of much interest. However, most complex network models in literatures limit only in description of one kind of parts and their one kind of interrelation of complex system.

Many problems involve several kinds of parts and interrelations. Then in terms of many existing complex network, one feasible approach is to composite them together so that multiple kinds of parts and interrelations could be studied naturally. However, simple connection styles, such as one-to-one correspondence, random connection or similar connection, are adopted. The interrelation which two networks establish would influence a lot on how they interconnect. In this study, a generalized complex network model of describing complex system involving several kinds of parts and interrelations is proposed. Through bringing the interrelations between parts into formalized definition of complex network, a four-tuple named multi-subnet composited complex network is given. Based on that, we propose reorganizations of complex networks compounding and reducing. Based on the model, transfer between bus and tube networks is studied by analyzing traffic dynamic on composited network. Three kinds of connection styles are discussed. And the properties of different kinds of nodes are also considered. Numerical simulations on artificial networks as well as real bus and tube networks of Qingdao in China agree well with our analysis, which shows validity of our model.

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